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13. ABSTRACT

This paper characterizes a class of constrained transportation problems (linear programming problems which are composed of a transportation problem with additional constraints) which can be transformed into equivalent transportation problems. An efficient computational procedure for finding an equivalent transportation problem is presented which introduces at most one origin and one destination for each extra constraint. Our results extend the procedures developed by Wagner and Manne for problems in which the extra constraints consist of bounding certain partial sums of variables.

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1. INTRODUCTION

Linear programming literature abounds with ingenious procedures for reformulating linear problems which seem unrelated to the transportation problem into an equivalent transportation problem [2,3,4,5,10,11,12]. Some of these linear problems are constrained transportation problems [7,8] (i.e., a linear programming problem consisting of a transportation problem with extra constraints). In this paper we substantially enlarge the class of constrained transportation problem formulations which have been shown to be transformable into an equivalent ordinary transportation problem by extending the procedures developed[1] by Wagner [10] and Manne [5, p. 382]. There are decided computational advantages to transforming a seemingly more general linear programming problem into a transportation problem when this is possible. For instance, the computational results of [1,6,9] disclose that transportation problems can be solved at least 100 times faster by special purpose codes than by general purpose linear programming methods. Thus, the difference between solving a constrained transportation problem with a general method and solving its counterpart transportation problem with a special purpose code should be very substantial, certainly not less than the factor of 100 to 1.

There are, of course, other advantages to transforming a constrained transportation problem into a transportation problem. By using the specialized algorithms [1,6,9], no round-off error is introduced. Also, if the total supply of the reformulated problem is not equal to the total demand, then one can conclude without further effort that the original problem lacks a feasible solution.

Wagner [10] and Manne [5, p. 382] showed that if an extra constraint to a transportation problem bounds a partial sum of variables in a single node constraint, then the problem can be reformulated as a larger transportation or network problem with additional nodes. This paper extends the techniques proposed by Wagner and Manne by demonstrating that certain extra constraints involving sums and/or differences of variables from several originsand/or destination constraints are in fact equivalent in the transportation setting to a bound on a partial sum of variables from a single origin or destination constraint. The coefficient structure of constraints in this class seems upon inspection to bear no relationship to the simpler equivalent forms. We present a general characterization for identifying these constraints and give a procedure for generating equivalent constraints that can be operated on by Wagner's or Manne's methods. Examples of problem classes in which the more general constraint structure appears are given in Section 4.

2. TRANSFORMABLE CONSTRAINTS AND THEIR EQUIVALENTS

The standard transportation problem can be stated as follows:

Minimize $\sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij}$

subject to

$$\sum_{j \in N} x_{ij} = a_i, \qquad i \in M,\ a_i \geq 0$$

$$\sum_{i \in M} x_{ij} = b_j, \qquad j \in N,\ b_j \geq 0$$

$$x_{ij} \geq 0$$

where $M = \{1,\ldots,m\}$ and $N = \{1,\ldots,n\}$ and $\sum_{i \in M} a_i = \sum_{j \in N} b_j$.

This problem is frequently presented in a tableau format as shown in Table 1 with a single row representing each origin constraint and a single column representing each destination constraint.

Our main assertion is that any singularly constrained transportation problem whose extra constraint is of the form

$$P: \sum_{i \in M'} \sum_{j \in J_i} \alpha_i x_{ij} \leq D$$

where $M' = M - \{p\}$ for some $p \in M$ and for each $i \in M'$, $J_i = S$ or $J_i = N-S$

and $\alpha_i = \begin{cases} +1 & \text{if } J_i = S \\ -1 & \text{if } J_i = N-S \text{ where } S \subset N \end{cases}$

can be transformed into an enlarged transportation problem. To give an example of one such constraint, consider a transportation problem with four origins and five destinations as depicted in Table 1. The associated supply and demand values are indicated on the right and lower borders. A specific constraint of the type P is

$$x_{21} + x_{23} + x_{25} + x_{31} + x_{33} + x_{35} - x_{42} - x_{44} \leq 9$$

whose non-zero coefficients appear in the table in the cells corresponding to the appropriate variables. In this example

$S = \{1,3,5\}$, $N-S = \{2,4\}$, $p = 1$, $J_2 = \{1,3,5\}$, $J_3 = \{1,3,5\}$, and $J_4 = \{2,4\}$.

	D_1	D_2	D_3	D_4	D_5	Supply
O_1						8
O_2	+1		+1		+1	7
O_3	+1		+1		+1	10
O_4		-1		-1		5
Demand	7	8	2	7	6	

Table 1

The procedure we propose for transforming such a constraint into a partial sum of variables in a single row of the transportation tableau creates row multiples R_i and column multiples K_j which generate a linear combination of the original transportation constraints. Thereupon, adding this linear combination to the type P constraint produces a constraint which can be accomodated by the methods of [5 ,10]. The procedure consists of a single iteration of the following instructions.

Step 1. Set $R_i = 0$ for all $i \varepsilon M$ and $K_j = 0$ for all $j \varepsilon N$.

Define $I_S = \{i \varepsilon M': \; J_i = S\}$

$I_{N-S} = \{i \varepsilon M': \; J_i = N-S\}$

Step 2. If $I_S = \emptyset$ go to step 3. Otherwise set $R_i = -1$ for all $i \varepsilon I_S$

Step 3. If $N-S = \emptyset$ go to step 4. Otherwise set $K_j = 1$ for all $j \varepsilon N-S$.

Step 4. Multiply the i^{th} origin and j^{th} destination constraints by their corresponding R_i and K_j values and sum to form a linear combination of the standard origin and destination constraints. The addition of the resulting equation to the constraint of type P produces an inequality bounding a sum of variables in row p.

To illustrate, we apply this procedure to the example in Table 1, Step 1 sets the multiples R_1, R_2, R_3, R_4, and K_1, K_2, K_3, K_4, and K_5 equal to zero and identifies the sets $I_S = \{2,3\}$ and $I_{N-S} = \{4\}$. In step 2, R_2 and R_3 are set equal to -1. In step 3, K_2 and K_4 are set equal to +1. The constraints of the transportation problem associated with non-zero R_i and K_j multiples and the extra constraint are shown below in detached coefficient form with the non-zero R_i and K_j multiples indicated on the left.

	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}		RHS
$R_2=(-1)$						1	1	1	1	1											=	7
$R_3=(-1)$											1	1	1	1	1						=	10
$K_2=(+1)$		1					1					1					1				=	8
$K_4=(+1)$				1					1					1					1		=	7
extra						1		1		1	1		1		1		-1			-1	$\leq$	9
		1		1																	$\leq$	7

Forming the linear combination described in step 4 and adding this to the extra constraint yields $x_{12} + x_{14} \leq 7$, as shown below the line. Using the techniques described by Wagner [10] and Manne [5, p. 382], the original problem can now be reformulated as an equivalent uncapacitated transportation problem with one additional origin and one additional destination.

The validity of the transformation procedure is established in the following theorem.

<u>Theorem</u>: The addition of a constraint of type P to the linear combination of the standard origin and destination constraints specified by the procedure yields an equivalent constraint bounding a sum of variables in row p.

<u>Proof</u>: The definitions of M', I_S, I_{N-S}, and δ_i give

$$0 \geq \sum_{i\in M'} \sum_{j\in J_i} \delta_i x_{ij} = \sum_{i\in I_S} \sum_{j\in S} x_{ij} - \sum_{i\in I_{N-S}} \sum_{j\in N-S} x_{ij}$$

Adding the appropriate linear combination of the transportation constraints to the expression on the right above yields

$$\sum_{i\varepsilon I_S}\sum_{j\varepsilon S}x_{ij}-\sum_{i\varepsilon I_{N-S}}\sum_{j\varepsilon N-S}x_{ij}-\sum_{i\varepsilon I_S}\sum_{j\varepsilon N}x_{ij}+\sum_{i\varepsilon M}\sum_{j\varepsilon N-S}x_{ij}$$

$$=-\sum_{i\varepsilon I_S}\sum_{j\varepsilon N-S}x_{ij}-\sum_{i\varepsilon I_{N-S}}\sum_{j\varepsilon N-S}x_{ij}\cdot\sum_{i\varepsilon M}\sum_{j\varepsilon N-S}x_{ij}$$

$$=-\sum_{i\varepsilon M'}\sum_{j\varepsilon N-S}x_{ij}+\sum_{i\varepsilon M}\sum_{j\varepsilon N-S}x_{ij}$$

$$=\sum_{i\varepsilon\{p\}}\sum_{j\varepsilon N-S}x_{ij}=\sum_{j\varepsilon N-S}x_{pj}\leq D'$$

where $D' = D + \sum_{i\varepsilon M} R_i a_i + \sum_{j\varepsilon N} K_j b_j = D - \sum_{i\varepsilon I_S} a_i + \sum_{j\varepsilon N-S} b_j$.

This completes the proof.

3. SOME EXTENSIONS

The general form of the constraint P is stated as a "less than or equal" constraint type, but the procedure is clearly valid for both equality and "greater than or equal" constraints. In the latter case, an inequality of the form $\sum_{j\varepsilon N-S} x_{pj} \geq D'$ will be produced, which may be made suitable for transformation by Wagner's and Manne's methods by noting that it is equivalent to $\sum_{j\varepsilon S} x_{pj} \leq a_p - D'$. Additionally, since the index sets M and N are arbitrarily associated with origins and destinations (or rows and columns), P could also describe a class of constraints whose members are equivalent to a partial sum of variables in a single column.

It should be clear that the δ_i values in constraint P could be any value so long as all δ_i are equal in absolute value. In this case step 2 would need to be altered to set the appropriate R_i to -k and in step 3 the K_j multiples should be set to +k where $|\delta_i| = k$ for $i \epsilon M'$. After finding the equivalent constraint in step 4 both sides of the constraint would then be multiplied through by 1/k to obtain unity coefficients for the variables. If the original right-hand side value of the extra constraint is an integer multiple of $|\delta_i|$ and if the a_i and b_j in the original transportation problem are integer valued, then the optimal solution to the equivalent enlarged transportation problem will be integer valued. Thus our procedure gives a convenient way to solve certain integer programming problems with the computationally efficient transportation algorithm.

If the original problem includes several constraints of type P then the procedure could be applied to each extra constraint independently. In this case if the transformed constraints involve disjoint sets of variables or if the sets of variables are nested in the same node constraint then an equivalent larger transportation problem can be derived via [5,10].

The presence of additional constraints necessarily raises questions of feasibility, and finding an equivalent restriction may shed light on the feasibility of the original problem. In particular, if the new constraint formed in step 4 results in all zero coefficients and a non-negative right-hand side then the extra constraint is redundant. Also, checking the sign of the coefficients and the sign of the right-hand side (in relation to the direction of the inequality) may indicate whether the problem has any feasible solution whatsoever.

When constraints can be identified as being of type P then the equivalent restriction can be found directly by the formula

$$\sum_{j \epsilon N-S} x_{pj} \leq D - k\sum_{i \epsilon I_S} a_i + k\sum_{j \epsilon N-S} b_j, \text{ where } k = |\delta_i| \text{ and}$$

the procedure of finding the appropriate linear combination need not be explicitly applied. This is, of course, a direct consequence of the proof of the theorem.

4. APPLICATIONS

Constraints of type P appear in many problems. For example, consider a problem where warehouses ship a perishable product to several markets and because several of the warehouses are physically distant from some of the markets a special container must be provided for each item shipped along certain routes. If the supply of these containers is limited then the problem can be modeled as a standard transportation problem with the additional container constraint. An example is illustrated in Table 4 where the +1 values denote the routes that require containers. The indicated constraint

	M_1	M_2	M_3	M_4
W_1				
W_2		+1		+1
W_3		+1		+1

Table 4.

is transformable and is equivalent to a constraint on the shipments from W_1 to M_1 and M_3.

A type of "balancing" constraint is also represented by the characterization P. Suppose that several warehouses can supply several markets. Given demands for the product and figures for warehouse supply a transportation model can be formulated to determine the minimum cost shipping schedule. If in addition management requires a certain degree of specialization, two constraints may be introduced to balance the shipments from certain warehouses to certain markets. Tables 5a and 5b represent two different balancing constraints when the right-hand side of both constraints is zero. The constraint indicated in Table 5c is equivalent to that in 5a, and the constraint in 5d is equivalent to that in 5b.(Since the constraints in Table 5c and in Table 5d include disjoint sets of variables, Wagner's procedure can be applied to incorporate both constraints in a larger uncapacitated transportation problem.)

Warehouse \ Market	1	2	3	4
A				
B			-1	-1
C	+1	+1		

Table 5a.

Warehouse \ Market	1	2	3	4
A			+1	+1
B	-1	-1		
C				

Table 5b.

Warehouse \ Market	1	2	3	4
A			+1	+1
B				
C				

Table 5c.

Warehouse \ Market	1	2	3	4
A				
B				
C	+1	+1		

Table 5d.

Balancing constraints or other constraints of type P also appear in models of trading between nations that seek to maintain favorable balances of trade by legal restrictions on their imports and exports. In such a model the total flow of goods from one nation may be required to differ by no more than a specified amount from the flow of goods in the reverse direction.

FOOTNOTES

[1] It has been brought to the authors' attention that these procedures were developed in linear programming courses by A. Charnes and W.W. Cooper at Carnegie-Mellon, Purdue, Northwestern, and the University of Texas.

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